A MODEL OF CAVITATIONAL FLOWS FOR INITIAL STAGES OF CAVITATION

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Initial stages of cavitation onset in a stream of liquid flowing past a body is considered. A certain pattern of hydrodynamic phenomena related to the onset of cavitational flows is porposed. It is based on the assumption of a comparatively high cavitation number and that a liquid with a relatively low content of gas-vapor bubbles moves within a certain zone.

Results of calculations are compared with experimentally determined dimensions of the cavitation zone. As an example, the flow past a step in a plane channel is considered.

1. Cases of erosion of concrete and metals due to cavitation are rather frequent in high-pressure hydraulic installations and in various hydraulic machines, when in the initial stage cavitation develops around protrusions of a smooth surface at comparatively high cavitation numbers:

$$\varkappa = \frac{2(p - p_v)}{\rho v^2} \tag{1.1}$$

Here p, ρ, and v are, respectively, the pressure, density, and velocity of the liquid, and p_V is the pressure of vapors of the latter.

In the potential flow of a perfect liquid around a certain body the lowest pressures are observed at the surface of the body, where, obviously, the onset of incipient cavitation is to be expected. Carefully conducted experiments had shown that, when an appropriate static pressure is reached, numerous vapor-gas bubbles appear in the region of minimum pressure on the surface of the body. Owing to the presence in ordinary water of sufficiently large cavitation nuclei, such bubbles are generated at pressures only slightly lower than that of saturated steam [1]. With decreasing pressure or increasing velocity of the oncoming stream the straining tension increases, thus creating favorable conditions for the appearance of ever smaller cavitation nuclei. At the same time the part of the body surface along which bubbles are generated increases in length.

The determination of the size of this part [of the body] makes it possible to estimate the size of the region in which erosion by cavitation will occur.

Photographs taken at very short time intervals show that the end of the cavitation region is pulsating. Its mean position is, however, well defined. A certain averaged motion is considered in the following.

A number of experimental investigations, and in particular those whose results are presented in [2], make it possible to conclude that in the flow past a body the pressure in the cavitation zone at the boundary of the body is constant. This boundary condition will be used in the following in the formulation of the boundary value problem.

Phenomena taking place in the cavitation region can be described as follows. In the immediate vicinity of a body velocity vectors of the liquid and of a certain imaginary medium consisting of evolving bubbles differ in magnitude and direction. The velocity and direction of motion of bubbles and liquid become subsequently equalized in a zone in which moves a mixture of liquid and bubbles. The latter may be considered as a kind of compressible fluid [3, 4]. In the initial stage of cavitation the motion is subsonic.

Let us assume that the width of the region of the "two-velocity" motion is small and can be considered as a kind of boundary layer. Hence, in the following and also when substituting the flow of a perfect fluid for that of a viscous one, we neglect this boundary layer and consider the whole of the region outside the body as a liquid with a comparatively low content of gas bubbles. The equation defining the motion of such fluid is of the elliptic kind, which for low subsonic velocities is similar to the Laplace equation.

When solving the problem of the cavitation zone length we use the Laplace equation as first approximation.

Comparison of calculated and experimental data shows these assumptions to be entirely acceptable, since they result in fairly accurate dimensions of the cavitation zone, whose size is of the greatest interest. This is to be expected since the size of this zone is to a certain extent an integral property.

On the above assumptions the boundary condition (based on the supposition of constant pressure in the cavitation zone) at the corresponding part of the body indicates the existence there of a nonzero normal component of velocity.

This results from a certain degree of approximation in our analysis. It should be noted that the occurrence of such fictitious divergent flows also appears in certain other models of cavitational flows (e. g., the Efros-Gilbarg model). The natural condition of continuity of pressure throughout the region of flow of liquid is used in the determination of the size of the cavitation zone.

We assume that the pressure on the part of the body where cavitation is present is constant, i. e., that the absolute velocity is constant, while along the remaining part the condition of streamline flow prevails, i. e., the direction of the velocity vector is specified.

Introducing in our analysis the function $\omega = \ln \zeta$, where ζ is the complex velocity, we obtain

$$\omega = \ln |\zeta| + i \arg \bar{\zeta} = u_1 + iv_1 \tag{1.2}$$

and for the determination of ω we thus have the conditions of the Hilbert problem. However, in the problem considered below it is more convenient to use the velocity-hodograph method.

When deriving further approximation it should be kept in mind that, if the modulus of velocity β and the angle θ of the velocity vector to the coordinate axis are expressed in new variables, the equation of motion of a compressible fluid reduces to linear equations of the Chaplygin type. In this case an exact solution can be derived for a body whose contour consists of straight line segments.

2. Consider the flow of a perfect fluid past a step at the bottom of a plane channel at the stage of incipient cavitation. Particular attention should be given to the investigation of the effect of the relative height h/b of the step (h is the height of the step and b that of the channel) and its geometry on the extent of the cavitation zone.

Assuming a uniform distribution of sufficiently small cavitation bubbles in the liquid, so that in the first approximation the latter can be considered to be isotropic and incompressible, we obtain for the velocity potential $\varphi(x, y)$ the Laplace equation $\Delta\varphi = 0$.

We introduce the plane of the complex variable $z = x + iy$ (Fig. 1). The problem of finding the size of the constant pressure region reduces to the construction of the complex flow potential $w(z) = \varphi + i\psi$, where $\psi(x, y)$ is the stream function.

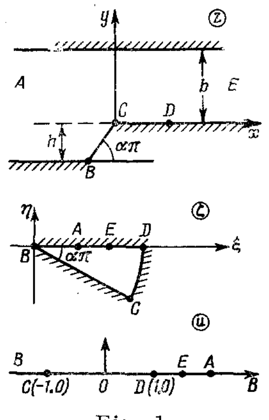

Fig. 1

The complex velocity

$$\zeta = dw / dz = \xi + i\eta$$

satisfies the following conditions: at an infinitely distant point A, $\zeta = v_A$, and along sections AE, AB, and DE we have $\eta = 0$. According to our assumptions the pressure along section CD is constant. Hence, from the Bernoulli integral

$$|\zeta|^2 = v_A^2 + 2(p - p_v)/\rho \text{ along CD}$$

From this we have, in particular,

$$v_C^2 = v_D^2 = v_A^2 + 2(p - p_v)/\rho$$

Thus in the hodograph plane ζ the region of the sector of angle $\alpha\pi$ $(0 < \alpha \leq 1/2)$ (Fig. 1) corresponds to the region of actual flow.

To solve this problem it is necessary to map the region occupied by the moving liquid onto this sector in the hodograph plane. First, we transfer these regions to the upper half-plane of the auxiliary variable u (Fig. 1) with the shown correspondence of points.

The function providing the conformal transformation of the upper half-plane of u onto the flow region in the z-plane is defined by the Christoffel-Schwartz integral

$$z = c_1 \int_{-1}^{u} \frac{(u+1)^\alpha \, du}{(u - u_A)(u - u_E)}$$

To find the constant of integration c_1 and the unknown coordinates of points A and E in plane u we make use of the fact that in circumventing these points in the u-plane along infinitely small half-circles, the function defined by integral (2.1) obtains increments $-(b + h)i$ and bi, respectively (Fig. 1).

We have two algebraic equations

$$c_1 \frac{(u_E + 1)^\alpha}{u_E - u_A} = \frac{b}{\pi}, \qquad c_1 \frac{(u_A + 1)^\alpha}{u_E - u_A} = \frac{b + h}{\pi} \tag{2.2}$$

Two of the unknowns in (2.2) can be expressed by the third, e. g., u_A, and the geometric dimensions of the channel:

$$u_E = (u_A + 1)\left(\frac{b}{b+h}\right)^{1/\alpha} - 1, \qquad c_1 = \frac{h}{\pi} \frac{u_A - u_E}{(u_A + 1)^\alpha - (u_E + 1)^\alpha} \tag{2.3}$$

Constant u_A will be determined later.

Integral (2.1) can be expressed in terms of elementary functions only for rational α. Hence in the following we assume $\alpha = m/n$, where m and n are positive integers.

Representing integral (2.1) as the sum of two integrals and introducing in each the substitution of variable of the form

$$(u + 1)^{1/n} = t \tag{2.4}$$

we obtain

$$z = \frac{c_1 n}{u_A - u_E}\left[(u_A + 1)\int_0^t \frac{t^{m-1}dt}{t^n - (u_A + 1)} - (u_E + 1)\int_0^t \frac{t^{m-1}dt}{t^n - (u_E + 1)}\right] \tag{2.5}$$

Let us examine the first integral in (2.5). A decomposition of the integrand into common fractions yields

$$\frac{t^{m-1}}{t^n - (u_A + 1)} = \frac{1}{n(u_A + 1)} \sum_{\nu=0}^{n-1} \frac{t_\nu^{\,m}}{t - t_\nu} \tag{2.6}$$

$$(t_\nu = |u_A + 1|^{1/n} e^{2\nu\pi i/n})$$

Each term of (2.6) is integrated from 0 to t with the principal value of the logarithm chosen as

$$\Lambda(t, t_\nu)=\int\limits_0^t \frac{dt}{t-t_\nu} = \ln\left(1-\frac{t}{t_\nu}\right) \tag{2.7}$$

We obtain the conformal mapping function z(u) in its final form as

$$z = \frac{c_1}{u_A - u_E}\left[\sum_{\nu=0}^{n-1} t_\nu^{\,m}\Lambda(t, t_\nu) - \sum_{\nu=0}^{n-1} s_\nu^{\,m}\Lambda(t, s_\nu)\right] \tag{2.8}$$

where s_ν are the roots of the equation

$$s^n - (u_E + 1) = 0$$

Using certain elementary functions, we map the region of the circular sector of the ζ-plane onto the upper half-plane u (Fig. 1):

$$u = \frac{1}{2}\left[\left(\frac{\zeta}{v_D}\right)^{1/\alpha} + \left(\frac{v_D}{\zeta}\right)^{1/\alpha}\right] \tag{2.9}$$

The coordinate of point A in the u-plane is found from (2.9) at $\zeta = v_A$:

$$u_A = \frac{1}{2}\left[\left(\frac{v_A}{v_D}\right)^{1/\alpha} + \left(\frac{v_D}{v_A}\right)^{1/\alpha}\right] \tag{2.10}$$

Having determined all of the constants appearing in the conformal mapping function (2.8), we find the coordinate of point D, equal in the z-plane to the length l of the constant pressure region, from (2.8) at $u = 1$ (Fig. 1):

$$l = \frac{c_1}{u_A - u_E}\left[\sum_{\nu=0}^{n-1}\left\{t_\nu^{\,m}\ln\left(1-\frac{2^{1/n}}{t_\nu}\right) - s_\nu^{\,m}\ln\left(1-\frac{2^{1/n}}{s_\nu}\right)\right\}\right] \tag{2.11}$$

In the particular case of a rectangular step at the bottom of the channel ($\alpha = 1/2$) formula (2.8) is considerably simplified:

$$z = \frac{h}{\pi\left(\sqrt{u_A+1} - \sqrt{u_E+1}\right)}\left(\sqrt{u_E+1}\ln\frac{\sqrt{u_E+1}+\sqrt{u+1}}{\sqrt{u_E+1}-\sqrt{u+1}} - \sqrt{u_A+1}\ln\frac{\sqrt{u_A+1}+\sqrt{u+1}}{\sqrt{u_A+1}-\sqrt{u+1}}\right) \tag{2.12}$$

Taking into account the relationship

$$v_D^2 = v_A^2(1+\varkappa)$$

we write the explicit expression for the dependence of l on the cavitation number $\varkappa$ at $\alpha = 1/2$:

$$l = \frac{b}{\pi}\ln\frac{[bf + (b+h)\sqrt{2}](f-\sqrt{2})}{[bf - (b+h)\sqrt{2}](f+\sqrt{2})} - \frac{h}{\pi}\ln\frac{f+\sqrt{2}}{f-\sqrt{2}} \tag{2.13}$$

where

$$f^2 = \frac{(2+\varkappa)^2}{2(\varkappa+1)}$$

Passing in formula (2.13) to the limit as $b \to \infty$, we obtain the length of the constant pressure region for the case of flow of an infinite stream past a rectangular step:

$$l_\infty = \frac{h}{\pi}\left[\frac{4(\varkappa+2)\sqrt{\varkappa+1}}{\varkappa^2} - 2\ln\frac{\sqrt{\varkappa+1}+1}{\sqrt{\varkappa+1}-1}\right] \tag{2.14}$$

The dependence of the relative length l/h of the constant pressure zone on the cavitation number $\varkappa$, calculated by formula (2.13) for several values of the relative height h/b of the step, is shown in Fig. 2. It was found that the height of a channel has a considerable effect on the cavitation zone length at h/b > 0.001.

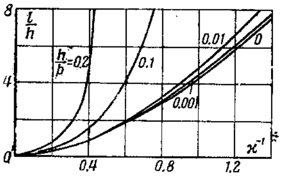

Fig. 2

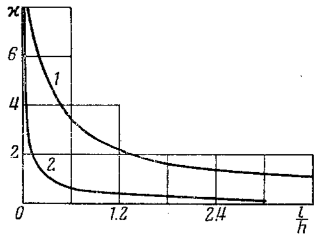

Fig. 3

Let us compare the length of the cavitation zone determined by formula (2.14) with the dimension of the zone in which the pressure is below that of the vapor of the liquid and which obtains in the solution of the problem of laminar flow of a fluid capable of withstanding infinitely great tensile stresses. In such cases the boundary condition $\eta = 0$ is satisfied along the entire length of half-line CE (Fig. 1).

The length l_∞^* of the section in which pressure $p < p_V$ is found in the usual way and is defined by the formula

$$l_\infty^* = \frac{h}{\pi}\left[\frac{2\sqrt{1+\varkappa}}{\varkappa} - \ln\frac{\sqrt{1+\varkappa}+1}{\sqrt{1+\varkappa}-1}\right] \qquad (2.15)$$

Functions $l_\infty^*(\varkappa)$ and $l_\infty^*(\varkappa)$ are shown in Fig. 3, where curve 1 relates to the length of the cavitation zone calculated by formula (2.14) and curve 2 to that of the region in which $p < p_v$ as defined by formula (2.15). It will be seen from Fig. 3 that for small $\varkappa$, $l_\infty \approx 10 l_\infty^*$, while for high cavitation numbers $l_\infty / l_\infty^* \to 2$.

3. A series of experimental investigations of cavitational flow past steps was carried out in the Cavitation Laboratory of the IPM (Institute of Applied Mechanics) of the Academy of Science of the USSR, with the participation of K. K. Shal'nev, with a view to confirming the assumptions on which the theoretical analysis was based.

The experiments were carried out in a hydrodynamic tube with a working section of 24 × 100 mm. Models of steps of height h = 9.0, 7.0, 3.6, and 1.8 mm where positioned in the constricted part of the working channel of 24 × 90 mm cross section. The relative height of steps was thus h/b = 0.10, 0.078, 0.040, and 0.020. Each model was tested at flow velocities in the range of 11–19 m/sec measured at the working chamber axis. Various stages of cavitation at v = const were produced by controlling the pressure in the tube.

The cavitation number was determined from velocity and pressure upstream of the step in the restricted section of the working chamber. The results were plotted in the form of curves of $\varkappa(\lambda)$, where $\lambda = l/h$ is the relative length of the cavitation zone and l the visually assessed length of that zone from the front edge of the step.

The length l of the cavitation zone was measured on a scale marked on the side of the step.

Certain stages of cavitation were photographed in the reflected light of a flash bulb at an exposition time of 1/2000 sec (Fig. 4). The photographs show that for λ = 0–5 the cavitation zone contains a mixture of liquid and vapor-gas bubbles. In the initial stages these bubbles are clustered at a certain distance from the front edge of the step in caverns which become periodically detached from the step. Such caverns are also observed in the proximity of the step edge itself. With decreasing cavitation number the zone structure becomes more homogeneous.

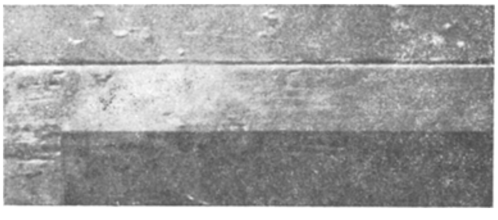

Fig. 4

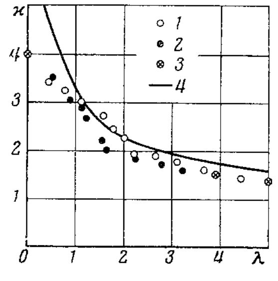

Fig. 5

The results of calculation of the relative length $\lambda = l/h$ of the cavitation zone by formula (2.13) for a step height $h = 9$ mm and velocity $v = 11.6$ m/sec are shown in Fig. 5, and in Fig. 6 for $h = 1.8$ mm and $v = 17.75$ m/sec. Experimental results are indicated in these figures by dots.

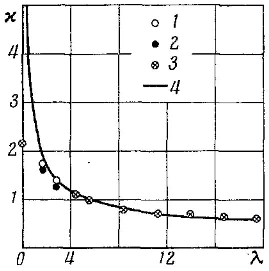

Fig. 6

The experimentally determined dependence $\varkappa(\lambda)$ shows a slight "hysteresis," depending on whether the experiment proceeds from suppressed cavitation to its ultimate separation stage (the direct process) or from the separation to the suppressed stage (the reverse process).

Experimental points lie close to the theoretical curves of $\varkappa(\lambda)$, except for values in the neighborhood of $\varkappa^*$ and corresponding to the instant of cavitation onset ($l = 0$).

The considerable effect on $\varkappa^*$ of the relative restriction h/b of the working channel is noticeable. This is to a certain extent explained by the velocity decrease in the boundary layer.

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